

ARITHMETICITY FOR PERIODS OF AUTOMORPHIC FORMS

WEE TECK GAN AND A. RAGHURAM

1. INTRODUCTION

Let G be a connected reductive algebraic group over a number field F , and let (π, V_π) be a cuspidal automorphic representation of $G(\mathbb{A}_F)$. Let H be an algebraic F -subgroup of G , and let χ be an automorphic character of $H(\mathbb{A}_F)$. We say that π has a non-vanishing (H, χ) -period if the functional

$$(1.1) \quad \phi \mapsto \ell_\chi(\phi) := \int_{[H]} \chi(h)^{-1} \phi(h) dh, \quad \phi \in V_\pi$$

is nonzero, where $[H] := H(F) \backslash H(\mathbb{A}_F)$ or sometimes $[H] := Z_G(\mathbb{A}_F) H(F) \backslash H(\mathbb{A}_F)$. Let us now suppose that we are in an arithmetic situation, in as much as that we can talk of the automorphic representation ${}^\sigma \pi$ for any $\sigma \in \text{Aut}(\mathbb{C})$. For example, if π is a cohomological cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$ then, by a result of Clozel (see Theorem 2.5 below), we know that so is ${}^\sigma \pi$. In this paper we study, mostly by the way of presenting a lot of examples, the dictum: *π has a non-vanishing (H, χ) -period if and only if ${}^\sigma \pi$ has a non-vanishing $(H, {}^\sigma \chi)$ -period.* It is this dictum that we call ‘arithmeticity for periods of automorphic forms.’

Let us remind the reader that automorphisms of \mathbb{C} , with the exceptions of the identity automorphism and complex-conjugation, are discontinuous; in particular, it is almost never the case that $\sigma(\ell_\chi(\phi)) = \ell_{\sigma \circ \chi}(\sigma(\phi))$. So one cannot naively take the σ inside the integral sign. In every example that we study, the dictum holds, and the argument is always indirect via some characterization of existence of such periods.

Let us also observe at the outset that the problem is a distinctly global problem. The corresponding local problem, at any finite place v , is trivial: if π_v is (H_v, χ_v) -distinguished, i.e., there exists a nonzero functional $\ell : \pi_v \rightarrow \mathbb{C}$ such that $\ell(\pi_v(h)v) = \chi_v(h)\ell(v)$ for all $h \in H(F_v)$. For any $\sigma \in \text{Aut}(\mathbb{C})$, it is easy to see that $\sigma \circ \ell$ gives a $(H_v, {}^\sigma \chi)$ -distinguishing functional for the conjugated representation ${}^\sigma \pi_v$.

The above local observation says that the problem of arithmeticity of automorphic periods is a consequence of a positive solution of the classical local-to-global problem: ‘If τ is an automorphic representation, and suppose at every place v , τ_v is (H_v, χ_v) -distinguished, then does τ have a nonzero global (H, χ) -period?’ Here, take τ to be ${}^\sigma \pi$.

Date: July 20, 2012.

1991 *Mathematics Subject Classification.*

W.T.G is partially supported by a startup grant at the National University of Singapore. A.R. is partially supported by the National Science Foundation (NSF), award number DMS-0856113, and an Alexander von Humboldt Research Fellowship.

Given a cuspidal automorphic representation π of $G(\mathbb{A})$, and a $\sigma \in \text{Aut}(\mathbb{C})$ we need to discuss when the representation ${}^\sigma\pi$ makes sense. This will be possible when the representation π contributes to the cohomology of a locally symmetric space of G with coefficients in a sheaf attached to a finite-dimensional coefficient system. In Section 2 we briefly discuss the appropriate cohomological preliminaries needed to talk about the Galois-conjugated representation ${}^\sigma\pi$.

In Section 3, we begin by looking at two of the easiest nontrivial examples when the ambient group $G = \text{GL}_2/F$. In particular, in the GL_2 context we look at the question of arithmeticity for Whittaker periods which boils down to every cuspidal automorphic representation being globally generic and that the space of cuspidal cohomology having a rational structure; indeed, the same ingredients give arithmeticity for Whittaker periods when $G = \text{GL}_n/F$. The other GL_2 example we analyze is (GL_1, χ) -periods for Hecke characters χ ; here, arithmeticity is a consequence of Hecke and Jacquet-Langlands Mellin transforms and Manin's and Shimura's classical algebraicity results on the critical values of L -functions for GL_2 .

In the rest of the paper we analyze the following situations for arithmeticity problems, which are various generalizations of the GL_2 cases considered in Section 3:

- (1) Shalika period integrals for representations of GL_{2n} . See Theorem 4.3. The nonvanishing of Shalika period integrals is characterized in terms of functorial transfers from $\text{GSpin}(2n+1)$.
- (2) $\text{GL}(n)/F$ -periods for representations of $\text{GL}(n)/E$, for a quadratic extension E/F . See Theorem 5.3. The nonvanishing of such period integrals is characterized in terms of functorial transfers from the unitary groups $\text{U}(n)$.
- (3) $\text{GL}(n-1)$ -periods for representations of $\text{GL}(n) \times \text{GL}(n-1)$. See Theorem 6.1. These are the Gross-Prasad periods for the general linear groups.
- (4) $\text{GL}(n) \times \text{GL}(n)$ -periods for representations of $\text{GL}(2n)$ over a totally real field.
- (5) Whittaker periods for classical groups, especially $\text{SO}(2n+1)$. See Theorem 9.1.
- (6) Gross-Prasad periods for classical groups, especially $\text{U}(n)$.

It is clear that these examples are pointing toward some general motivic interpretation of period integrals. Automorphic representations with a nonzero (H, χ) -period are usually characterized in terms of functorial transfers and/or in terms of some L -function attached to π having a pole or (not having) a zero at a certain point. In terms of L -values, the situation is very similar to a conjecture of Gross on motivic L -functions; see [9, Conjecture 2.7 (ii)]. This says that for a critical motive M , the order of vanishing of the critical L -value $L(\sigma, M, 0)$ is independent of the conjugating automorphism σ . In our situation, suppose π corresponds to a motive, and suppose having a non-vanishing (H, χ) -period corresponds to the (non-)vanishing of an L -value attached to π which happens to be a critical L -value, then Gross's conjecture would predict the validity of the dictum. For example, the situation in (3), respectively (4), above exactly ties up with critical L -values of the underlying Rankin-Selberg L -function, respectively the standard L -function.

Acknowledgements: The authors thank Jeff Adams for several helpful discussions concerning the proof of Theorem 9.1. It is also a pleasure to thank Dipendra Prasad, C.S. Rajan, A. Sankaranarayanan and Jyoti Sengupta for organizing a very memorable international colloquium at TIFR during which this article took shape. This article was completed during the first author's visit at the IHES at Bures-sur-Yvette in July 2012; the first author thanks the IHES for its support and for providing a peaceful yet stimulating working environment.

2. SOME COHOMOLOGICAL PRELIMINARIES

We will often talk about a 'cohomological cuspidal automorphic representation'. In this section, we will briefly review its definition and discuss some of its very basic properties that we need later.

Let G/\mathbb{Q} be a connected split reductive algebraic group over \mathbb{Q} and Z/\mathbb{Q} the center of G and let S be the maximal \mathbb{Q} -split torus in Z . Let C_∞ be a maximal compact subgroup of $G(\mathbb{R})$ and let $K_\infty = C_\infty S(\mathbb{R})$. The connected component of the identity of K_∞ is denoted K_∞° . For any open-compact subgroup $K_f \subset G(\mathbb{A}_f)$ define the space

$$S_{K_f}^G := G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty^\circ K_f.$$

Let $\mu \in X^+(T)$ be a dominant integral weight and E_μ be the algebraic irreducible representation of $G(\mathbb{C})$ with highest weight μ . Let \mathcal{E}_μ be the associated sheaf on $S_{K_f}^G$. We are interested in the sheaf cohomology groups

$$H^\bullet(S_{K_f}^G, \mathcal{E}_\mu)$$

on which there is a natural action of a suitable Hecke algebra.

We can compute the above sheaf cohomology via the de Rham complex, and then reinterpreting the de Rham complex in terms of the complex computing relative Lie algebra cohomology, we get the isomorphism:

$$H^\bullet(S_{K_f}^G, \mathcal{E}_\mu) \simeq H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; C^\infty(G(\mathbb{Q}) \backslash G(\mathbb{A}))^{K_f} \otimes E_\mu).$$

The inclusion $C_{\text{cusp}}^\infty(G(\mathbb{Q}) \backslash G(\mathbb{A})) \hookrightarrow C^\infty(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ of the space of smooth cusp forms in the space of all smooth functions induces, via well-known results of Borel [5], an injection in cohomology; this defines cuspidal cohomology:

$$\begin{array}{ccc} H^\bullet(S_{K_f}^G, \mathcal{E}_\mu) & \longrightarrow & H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; C^\infty(G(\mathbb{Q}) \backslash G(\mathbb{A}))^{K_f} \otimes E_\mu) \\ \uparrow & & \uparrow \\ H_{\text{cusp}}^\bullet(S_{K_f}^G, \mathcal{E}_\mu) & \longrightarrow & H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; C_{\text{cusp}}^\infty(G(\mathbb{Q}) \backslash G(\mathbb{A}))^{K_f} \otimes E_\mu) \end{array}$$

Using the usual decomposition of the space of cusp forms into a direct sum of cuspidal automorphic representations, we get the following fundamental decomposition

$$(2.1) \quad H_{\text{cusp}}^\bullet(S_{K_f}^G, \mathcal{E}_\mu) = \bigoplus_{\Pi} H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; \Pi_\infty \otimes E_\mu) \otimes \Pi_f^{K_f}$$

of $\pi_0(G_\infty) \times C_c^\infty(G(\mathbb{A}_f) // K_f)$ -modules.

We say that Π is a *cohomological cuspidal automorphic representation* if Π has a nonzero contribution to the above decomposition for some μ and some K_f . Equivalently, if Π is a cuspidal automorphic representation whose representation at infinity Π_∞ after twisting by E_μ has nontrivial relative Lie algebra cohomology. In this situation, we write $\Pi \in \text{Coh}(G, \mu)$, suppressing the level structure K_f .

One may consider cohomology with compact supports $H_c^\bullet(S_{K_f}^G, \mathcal{E}_\mu)$. *Inner cohomology* is defined as the image of compactly supported cohomology in global cohomology:

$$H_!^\bullet(S_{K_f}^G, \mathcal{E}_\mu) := \text{Image} \left(H_c^\bullet(S_{K_f}^G, \mathcal{E}_\mu) \longrightarrow H^\bullet(S_{K_f}^G, \mathcal{E}_\mu) \right).$$

It is a fundamental fact (which comes from analyzing the long-exact sequence arising from the Borel-Serre compactification; see, for example, Schwermer [40]) that

$$(2.2) \quad H_{\text{cusp}}^\bullet(S_{K_f}^G, \mathcal{E}_\mu) \subset H_!^\bullet(S_{K_f}^G, \mathcal{E}_\mu).$$

On the other hand, since any compactly supported function is square integrable, we also have that inner cohomology sits inside those cohomology classes which are represented by square-integrable automorphic forms, i.e.

$$(2.3) \quad H_!^\bullet(S_{K_f}^G, \mathcal{E}_\mu) \subset H_{(2)}^\bullet(S_{K_f}^G, \mathcal{E}_\mu).$$

Let us now briefly recall the action of $\text{Aut}(\mathbb{C})$ on algebraic cuspidal representations Π of $G(\mathbb{A}_F)$. (For more details, see Clozel [6].) Suppose $\Pi = \Pi_\infty \otimes \Pi_f$ be the decomposition of Π into its infinite and finite parts. Let W be the representation space of the finite part Π_f . Choose any σ -linear isomorphism $t : W \rightarrow W'$, and define the representation ${}^\sigma\Pi_f$ by

$${}^\sigma\Pi_f(g_f) = t \circ \Pi_f(g_f) \circ t^{-1}.$$

Up to equivalence of representations, the definition of ${}^\sigma\Pi_f$ is independent of the choices W' and t . Next, suppose $\Pi_\infty = \otimes_{v \in S_\infty} \Pi_v$; here S_∞ is the set of infinite places of F . Suppose F is totally real, then S_∞ may be identified with the set of all embeddings of F into \mathbb{C} (indeed, into \mathbb{R}), and we define

$${}^\sigma\Pi_\infty = \otimes_{v \in S_\infty} \Pi_{\sigma^{-1} \circ v}.$$

The definition of ${}^\sigma\Pi_\infty$ when F is any number field is a little more involved, and we refer the reader to Clozel [6, Definition 3.6 on p.107]. Putting the finite and infinite parts together, we have:

$${}^\sigma\Pi = {}^\sigma\Pi_\infty \otimes {}^\sigma\Pi_f.$$

It is, *a priori*, only an abstract irreducible representation of the adelic group $G(\mathbb{A}_F)$. However, we have:

Proposition 2.4. *Suppose that π is a cohomological cuspidal automorphic representation of $G(\mathbb{A}_F)$. Then for any $\sigma \in \text{Aut}(\mathbb{C})$, there exists an automorphic representation τ_σ appearing in the automorphic discrete spectrum of $G(\mathbb{A}_F)$ such that ${}^\sigma\pi_f = \tau_{\sigma, f}$.*

Proof. By assumption, π_f appears as a Hecke-summand in H_{cusp}^\bullet . Since inner cohomology is sheaf-theoretically and hence rationally defined, we deduce by (2.2) and (2.3) that ${}^\sigma\pi_f$ appears in $H_!^\bullet$ and hence in $H_{(2)}^\bullet(S_{K_f}^G, \mathcal{E}_\mu)$. This proves the proposition. \square

When $G = \mathrm{GL}(n)$, one has the following stronger result:

Theorem 2.5. *If Π is a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ the representation ${}^\sigma\Pi$ is also a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$.*

In the classical situation of Hilbert modular forms see Shimura [43, Section 2]. For representations of GL_2 , see Harder [20] and Waldspurger [46], and more generally, for $\mathrm{GL}_n(\mathbb{A}_F)$ see Clozel [6, Théorème 3.13]. In Section 8, we shall consider the possibility of a similar strengthening of Proposition 2.4 for the classical groups.

3. THE GL_2 -EXAMPLES

Let us, for the sake of simplicity, take $G = \mathrm{GL}_2/\mathbb{Q}$, although everything discussed in this section works for GL_2 over any number field.

3.1. Whittaker periods. For the subgroup H we take

$$H = U_2 = U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{G}_a \right\},$$

i.e., U is the unipotent radical of the standard Borel subgroup of upper triangular matrices in G . Fix a nontrivial additive character $\psi : \mathbb{Q} \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$. Then, as usual, ψ gives a character $\psi : U(\mathbb{Q}) \backslash U(\mathbb{A}) \rightarrow \mathbb{C}^\times$ by $\psi \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) = \psi(x)$. Using the same symbol ψ for both the characters will cause no confusion. In this situation, the linear functional ℓ_ψ defined in (1.1) is called a global Whittaker functional.

Given a cuspidal automorphic representation (π, V_π) of $\mathrm{GL}_2(\mathbb{A})$ we can define for each $\phi \in V_\pi$ the associated Whittaker vector

$$(3.1) \quad W_\phi(g) := \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} \phi(ug) \psi(u)^{-1} du.$$

Observe that $W_\phi(1) = \ell_\psi(\phi)$. Using the action of $\mathrm{GL}_2(\mathbb{A})$ we see that $\ell_\psi(\phi) \neq 0$ for some ϕ if and only if $W_\phi \neq 0$ for some ϕ . A fundamental fact at the heart of the GL_2 -theory of automorphic forms is that W_ϕ determines ϕ . (See, for example, [7, Lecture 4, Section 1].) Indeed we have a Fourier expansion of the form

$$(3.2) \quad \phi(g) = \sum_{\gamma \in \mathbb{Q}^\times} W_\phi \left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right),$$

In particular, every cuspidal automorphic representation has a nonvanishing Whittaker period.

Now let us suppose that π is a cohomological cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$, and in particular, π has nonvanishing Whittaker periods. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$ Theorem 2.5 says that ${}^\sigma\pi$ is also a (cohomological) cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. Hence, by the above discussion once again, ${}^\sigma\pi$ also has nonvanishing Whittaker periods, i.e., we have arithmeticity for Whittaker periods for GL_2 .

The main ingredients in arithmeticity for Whittaker periods are (3.2) and Theorem 2.5. Both these ingredients, which are nontrivial assertions, are valid for GL_n/F over any number

field F after suitable modification; for example, the Fourier expansion takes the form:

$$(3.3) \quad \phi(g) = \sum_{\gamma \in \mathrm{GL}_{n-1}(F)/U_{n-1}(F)} W_\phi \left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right),$$

(See, for example, [7, loc.cit.].) Hence we get arithmeticity for Whittaker periods for GL_n/F . In Section 9 we consider the case of classical groups, especially split $\mathrm{SO}(2n+1)$, where the analysis is far more complicated.

Reverting to GL_2/\mathbb{Q} , let us go through the analysis for arithmeticity for Whittaker periods in the classical context of modular forms. Fix a positive integer N and consider the space $S_k(N)$ consisting of all holomorphic cusp forms of weight k on the upper half plane for the discrete subgroup $\Gamma_1(N)$ of $\mathrm{SL}_2(\mathbb{R})$. A cuspform $\varphi \in S_k(N)$ has a Fourier expansion

$$\varphi(z) = \sum_{n=1}^{\infty} a_n(\varphi) e^{2\pi i n z}.$$

Now define $S_k(N, \mathbb{Q})$ to be the \mathbb{Q} -subspace of the \mathbb{C} -vector space $S_k(N)$ consisting of all φ such that $a_n(\varphi) \in \mathbb{Q}$ for all $n \geq 1$. One has the following *nontrivial* fact:

$$(3.4) \quad S_k(N) = S_k(N, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

(See, for example, Shimura [44, Theorem 3.52].) This may be stated as the fact that the space of cusp forms of weight k and level N has a basis of cusp forms all of whose Fourier coefficients are in \mathbb{Q} . Indeed, there is a deeper integrality statement which says that the above is true with \mathbb{Z} instead of \mathbb{Q} ; however, for our purposes, a \mathbb{Q} -basis is sufficient. Let us note that (3.4) is the classical analogue of the statement (see Clozel [6, Théorème 3.19]) that cuspidal cohomology for GL_n/F admits a suitable rational structure. Now, given $\varphi \in S_k(N)$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$ we can define a function ${}^\sigma\varphi$ via a q -expansion.

$${}^\sigma\varphi(z) := \sum_{n=1}^{\infty} \sigma(a_n(\varphi)) e^{2\pi i n z}.$$

It follows from (3.4) that ${}^\sigma\varphi \in S_k(N)$. This is the classical analogue of Theorem 2.5. Arithmeticity for Whittaker models takes the form:

$$a_n(\varphi) \neq 0 \implies a_n({}^\sigma\varphi) \neq 0,$$

which is built into the definition of the Galois conjugate ${}^\sigma\varphi$. The depth of the phenomenon lies in the rationality statement in (3.4).

3.2. GL_1 -periods. We continue with $G = \mathrm{GL}_2/\mathbb{Q}$ and now we take

$$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} : x \in \mathbb{G}_m \right\} \simeq \mathrm{GL}_1.$$

Take a Hecke character $\chi : \mathbb{Q}^\times \backslash \mathbb{A}^\times \rightarrow \mathbb{C}^\times$, which gives a character $\chi : H(\mathbb{Q}) \backslash H(\mathbb{A}) \rightarrow \mathbb{C}^\times$ by $\chi \left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right) = \chi(x)$. Using the same symbol χ for both the characters will cause no confusion. Consider a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$. Suppose there is a $\phi \in V_\pi$ such that

$$\ell_\chi(\phi) = \int_{x \in \mathbb{Q}^\times \backslash \mathbb{A}^\times} \phi \left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right) \chi(x) dx \neq 0.$$

To analyze these integrals, and to relate them to L -values, following Jacquet-Langlands [23], fix a nontrivial additive character ψ as in the previous subsection and consider the Whittaker model $\mathcal{W}(\pi) = \mathcal{W}(\pi, \psi)$ of π . Let the cusp form ϕ correspond to $W_\phi \in \mathcal{W}(\pi)$ where W_ϕ is defined in (3.1). Then for a complex variable s such that $\Re(s) \gg 0$ the classical unfolding argument gives:

$$\ell(s, \phi, \chi) := \int_{x \in \mathbb{Q}^\times \backslash \mathbb{A}^\times} \phi \left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right) \chi(x) |x|^s dx = \int_{x \in \mathbb{A}^\times} W_\phi \left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right) \chi(x) |x|^s dx.$$

Denote the *zeta integral* on the right hand side by $Z(s, W_\phi, \chi)$, and note that all the ingredients in that integral are factorizable. Changing notation if necessary, there is a cusp form ϕ so that $\ell_\chi(\phi) \neq 0$ and the associated Whittaker vector W_ϕ is a pure-tensor $W_\phi = \otimes W_p$. Outside a finite set of primes S containing the infinite prime and all the primes where π or χ is ramified, one knows that W_p is the spherical vector and the local zeta-integral computes the local L -function:

$$Z(s, W_p, \chi_p) = \int_{x_p \in \mathbb{Q}_p^\times} W_p \left(\begin{pmatrix} x_p & 0 \\ 0 & 1 \end{pmatrix} \right) \chi_p(x_p) |x_p|_p^s dx_p = L(s, \pi_p \otimes \chi_p).$$

Let $L^S(s, \pi \otimes \chi) := \prod_{p \notin S} L_p(s, \pi_p \otimes \chi_p)$ denote the partial L -function. So far we have:

$$\ell(s, \phi, \chi) = Z(s, W_\phi, \chi) = \left(\prod_{p \in S} Z(s, W_p, \chi_p) \right) \cdot L^S(s, \pi \otimes \chi).$$

Now multiply and divide the right hand side by the local L -factors at $p \in S$ to get:

$$(3.5) \quad \ell(s, \phi, \chi) = \left(\prod_{p \in S} \frac{Z(s, W_p, \chi_p)}{L_p(s, \pi_p \otimes \chi_p)} \right) \cdot L(s, \pi \otimes \chi)$$

The integral $\ell(s, \phi, \chi)$ converges for all s since ϕ is rapidly decreasing. On the right hand side, one knows from Jacquet-Langlands that each of the ratios $Z(s, W_p, \chi_p)/L_p(s, \pi_p \otimes \chi_p)$, *a priori* defined only for $\Re(s) \gg 0$, in fact have an analytic continuation to all of s (see, for example, [14, Theorem 6.12 (ii)]), and that the completed L -function $L(s, \pi \otimes \chi)$ is an entire function of s (see, for example, [14, Theorem 6.18]).

We can now prove the following characterization of existence of (GL_1, χ) -periods and nonvanishing of a certain L -value:

Proposition 3.6. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$, and χ a Hecke character of \mathbb{Q} . Then, the following are equivalent:*

- (1) *There exists a cusp form $\phi \in V_\pi$ such that $\ell_\chi(\phi) \neq 0$.*
- (2) *$L(\frac{1}{2}, \pi \otimes \chi) \neq 0$.*

Proof. For (i) \Rightarrow (ii), put $s = 1/2$ in (3.5) to get:

$$0 \neq \ell_\chi(\phi) = r \cdot L(\frac{1}{2}, \pi \otimes \chi)$$

where r is an ad-hoc notation for the product $\prod_{p \in S} Z(\frac{1}{2}, W_p, \chi_p)/L_p(\frac{1}{2}, \pi_p \otimes \chi_p)$. Hence the right hand side is not zero.

For (ii) \Rightarrow (i), given $L(\frac{1}{2}, \pi \otimes \chi) \neq 0$, to construct a cusp form ϕ with non-vanishing period, we construct the associated Whittaker vector W_ϕ as a pure-tensor. Outside a finite set S as above, take W_p to be the normalized spherical vector. For places in S , given any s_0 (for us $s_0 = 1/2$), we are always guaranteed the existence of a Whittaker vector W_p such that the ratio $Z_p(s_0, W_p, \chi_p)/L(s_0, \pi_p \otimes \chi_p) \neq 0$. See [14, (6.29)]. (**Note:** Indeed, for GL_2 there is a W_p for each place p so that the local zeta integral computes the local L -factor, and so the ratio is in fact 1. We deliberately stated it in a weaker form of just nonvanishing of that ratio as that is the way it will generalize to $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$.) Now put all the local Whittaker vectors to get a global Whittaker vector, and take ϕ to be the associated cusp form. The proof follows again from (3.5) at $s = 1/2$. \square

Remark 3.7. Observe that it is possible for $L(\frac{1}{2}, \pi \otimes \chi) \neq 0$ and yet $L_f(\frac{1}{2}, \pi \otimes \chi) = 0$. (We use $L(s, \dots)$ for the completed L -function, and $L_f(s, \dots)$ for the finite part.) Such a phenomenon will happen when the infinite part $L_\infty(s, \pi \otimes \chi)$ has a pole at $s = 1/2$. Here is an easy example: Let $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ be the Ramanujan Δ -function which is a weight 12 cusp form of full level. Let $\pi := \pi(\Delta) \otimes |\cdot|^{-6}$ and take χ to be the trivial character. (For us, cuspidal automorphic representations need not be unitary, and indeed, π is not unitary.) Anyway, let $L(s, \pi)$ be the Jacquet-Langlands L -function, and $L(s, \Delta)$ be the classical Hecke L -function; then $L(s, \pi) = L(s - 6, \pi(\Delta)) = L(s - 1/2, \Delta)$. Using the classical functional equation $L(s, \Delta) = L(12 - s, \Delta)$ we get

$$L(\frac{1}{2}, \pi) = L(0, \Delta) = L(12, \Delta) \neq 0,$$

The L -factor at infinity is given by:

$$L_\infty(s, \pi_\infty) = L_\infty(s - 6, \pi(\Delta)_\infty) = 2(2\pi)^{-s+\frac{1}{2}} \Gamma(s - \frac{1}{2})$$

(For the last equation, see, for example, [38, 4.4]; the presence of an additional factor of 2 makes no difference to the discussion at hand.) Hence, $L_\infty(s, \pi_\infty)$ has a pole at $s = 1/2$, in other words, *nonvanishing of the global L -function at a (seemingly interesting) point does not guarantee that that point is a critical point.*

Now, given a cuspidal representation π as above with a nonvanishing (GL_1, χ) -period, and consequently with $L(\frac{1}{2}, \pi \otimes \chi) \neq 0$, we want to analyze the dictum of arithmeticity, whence we take π to be of cohomological type. But, even if π is of cohomological type and with $L(1/2, \pi \otimes \chi) \neq 0$ does not guarantee that $s = 1/2$ is a critical value. The same counterexample as in the above remark will work for this. Henceforth, we assume in addition that $s = 1/2$ is a critical point; i.e., by definition, $L_\infty(s, \pi \otimes \chi)$ and $L_\infty(s, \pi^\vee \otimes \chi^{-1})$ are regular at $s = 1/2$. Since local L -values are always nonzero, under the additional assumption of criticality of $s = 1/2$, we get

$$(3.8) \quad L(\frac{1}{2}, \pi \otimes \chi) \neq 0 \iff L_f(\frac{1}{2}, \pi \otimes \chi) \neq 0.$$

Now one can prove arithmeticity, for which we need the following algebraicity theorem due to Manin [33] in certain special cases, and more generally due to Shimura [42]; for the version stated below, see [38].

Proposition 3.9. *Let π be a cohomological cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. There exists two nonzero complex number $p^\pm(\pi)$ such that if $s = \frac{1}{2}$ is critical then for any*

algebraic Hecke character χ , and any $\sigma \in \text{Aut}(\mathbb{C})$ we have

$$\sigma \left(\frac{L_f(\frac{1}{2}, \pi \otimes \chi)}{p^{\epsilon_\chi(\pi)} \mathcal{G}(\chi) (2\pi i)^{d_\infty}} \right) = \frac{L_f(\frac{1}{2}, {}^\sigma \pi \otimes {}^\sigma \chi)}{p^{\epsilon_\chi(\pi)} \mathcal{G}({}^\sigma \chi) (2\pi i)^{d_\infty}},$$

where $\mathcal{G}(\chi)$ is the Gauß sum attached to χ , ϵ_χ is a sign keeping track of the parity of χ , and d_∞ is an integer determined entirely by π_∞ . (For more details see [38].)

A trivial corollary to the above deep proposition is that

$$(3.10) \quad L_f(\frac{1}{2}, \pi \otimes \chi) \neq 0 \iff L_f(\frac{1}{2}, {}^\sigma \pi \otimes {}^\sigma \chi) \neq 0.$$

The reader should compare this with Gross's conjecture mentioned in the introduction.

Theorem 3.11 (Arithmeticity for (GL_1, χ) -periods for GL_2). *Let π be a cohomological cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_\mathbb{Q})$. Suppose that π has a nonvanishing (GL_1, χ) -period for an algebraic Hecke character of \mathbb{Q} . Suppose, further, that $s = 1/2$ is a critical point for the L -function $L(s, \pi \otimes \chi)$. Then ${}^\sigma \pi$ has a nonvanishing $(\text{GL}_1, {}^\sigma \chi)$ -period.*

Proof. Follows from Proposition 3.6, (3.8) and (3.10). \square

Before closing this section, let us note that the above discussion is equivalent to taking:

$$G = \text{GL}_2 \times \text{GL}_1, \quad \text{and} \quad H = \Delta \text{GL}_1 := \left\{ \left(x, \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right) : x \in \text{GL}_1 \right\}.$$

It is from this perspective that it generalizes readily to the context of GL_n and GL_{n-1} which we discuss in Section 6.

4. ARITHMETICITY OF SHALIKA MODELS FOR GL_{2n}

In the remainder of the paper, we shall consider various generalizations of the results in the previous section. We will now define the notion of a *Shalika model* of a cuspidal automorphic representation Π of $\text{GL}_{2n}(\mathbb{A})$ where \mathbb{A} is the adèle ring of a number field F ; this particular situation was our original motivation to consider arithmeticity questions for periods. Let

$$\mathcal{S} := \left\{ s = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \mid \begin{matrix} h \in \text{GL}_n \\ X \in \text{M}_n \end{matrix} \right\} \subset \text{GL}_{2n} =: G.$$

It is traditional to call \mathcal{S} the Shalika subgroup of G . A characters $\eta : F^\times \backslash \mathbb{A}^\times \rightarrow \mathbb{C}^\times$ and a character $\psi : F \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$ can be extended to a character of $\mathcal{S}(\mathbb{A})$:

$$s = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \mapsto (\eta \otimes \psi)(s) := \eta(\det(h)) \psi(\text{Tr}(X)).$$

We will also denote $\eta(s) = \eta(\det(h))$ and $\psi(s) = \psi(\text{Tr}(X))$. For a cusp form $\varphi \in \Pi$, suppose $\eta^n = \omega_\Pi$ we can consider the integral

$$\mathcal{S}_\psi^\eta(\varphi)(g) := \int_{Z_G(\mathbb{A}) \mathcal{S}(F) \backslash \mathcal{S}(\mathbb{A})} (\Pi(g) \cdot \varphi)(s) \eta^{-1}(s) \psi^{-1}(s) ds, \quad g \in \text{GL}_{2n}(\mathbb{A}).$$

When $n = 1$, observe that \mathcal{S}_ψ^η is simply the Whittaker period of $\text{GL}(2)$, since η is forced to be the central character of Π .

The following theorem, due to the works of many people (Jacquet–Shalika [25], Asgari–Shahidi [3, 4], Hundley–Sayag [22]) gives a necessary and sufficient condition for \mathcal{S}_ψ^η to be non-zero.

Theorem 4.1 (Shalika Models). *Let Π be a cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$. For a pair of characters (η, ψ) , the following are equivalent:*

- (i) *There is a $\varphi \in \Pi$ and $g \in G(\mathbb{A})$ such that $\mathcal{S}_\psi^\eta(\varphi)(g) \neq 0$.*
- (ii) *\mathcal{S}_ψ^η defines an injection of $G(\mathbb{A})$ -modules*

$$\Pi \hookrightarrow \mathrm{Ind}_{\mathcal{S}(\mathbb{A})}^{G(\mathbb{A})}[\eta \otimes \psi].$$

- (iii) *Let S be any finite set of places containing $S_{\Pi, \eta}$. The twisted partial exterior square L -function*

$$L^S(s, \Pi, \wedge^2 \otimes \eta^{-1}) := \prod_{v \notin S} L(s, \Pi_v, \wedge^2 \otimes \eta_v^{-1})$$

has a pole at $s = 1$.

- (iv) *Π is the transfer of a globally generic cuspidal automorphic representation π of $\mathrm{GSpin}_{2n+1}(\mathbb{A})$ whose central character $\omega_\pi = \eta$.*

If Π satisfies any one, and hence all, of the equivalent conditions of Theorem 4.1, then we say that Π has an (η, ψ) -Shalika model, and we call the isomorphic image $\mathcal{S}_\psi^\eta(\Pi)$ of Π under \mathcal{S}_ψ^η a global (η, ψ) -Shalika model of Π .

One has a local analog of Theorem 4.1 which is due to Jiang-Nien-Qin [26]:

Theorem 4.2. *Let v be a non-archimedean place of F and suppose that Π_v is a supercuspidal representation of $\mathrm{GL}_{2n}(F_v)$ with L -parameter ϕ_v . Then the following are equivalent:*

- (i) *Π_v has a local (η_v, ψ_v) -Shalika model.*
- (ii) *The local twisted exterior square L -function $L(s, \wedge^2 \phi_v \otimes \eta_v^{-1})$ has a pole at $s = 0$.*
- (iii) *The L -parameter ϕ_v factors through the subgroup $\mathrm{GSp}_{2n}(\mathbb{C})$ with similitude character η_v .*

To be honest, this theorem was shown in [26] only when η_v is trivial, but the methods of their proof can be extended to the case of general η_v . Ideally, one would like to have the above theorem for discrete series representations. In this direction, we mention the paper [30] of Kewat.

Finally, here is the main result of this section:

Theorem 4.3. (Arithmeticity of Shalika periods.) *Let Π be a cohomological cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ which has an (η, ψ) -Shalika model. Assume that one of the following conditions holds:*

- (1) *The number field F is totally real; or*
- (2) *there is a finite place v where Π_v is supercuspidal.*

Then, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, ${}^\sigma \Pi$ is a cohomological cuspidal automorphic representation with a $({}^\sigma \eta, \psi)$ -Shalika model.

Proof. The proof, by the first author, is the content of the appendix of [18]; but for the sake of completeness we briefly sketch the proof. To begin, let us suppose that F is totally real.

Since Π has (η, ψ) -Shalika model, it follows by Thm. 4.1 that $L^S(s, \Pi, \wedge^2 \otimes \eta^{-1})$ has a pole at $s = 1$, and thus $\Pi^\vee \cong \Pi \otimes \eta^{-1}$. Now, we note the following:

- By recent results of Asgari–Shahidi [3, 4] and Hundley–Sayag [22], Π is a Langlands functorial transfer of a cuspidal representation of $\mathrm{GSpin}_{2n+1}(\mathbb{A})$ with central character η . Moreover, the lift is strong at the archimedean places, i.e., for each archimedean place, the L-parameter ϕ_v of Π_v factors through the dual group $\mathrm{GSp}_{2n}(\mathbb{C})$ of GSpin_{2n+1} with similitude character η_v .
- For any $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$${}^\sigma \Pi^\vee \cong {}^\sigma \Pi \otimes {}^\sigma \eta^{-1},$$

and thus

$$L^S(s, {}^\sigma \Pi \otimes {}^\sigma \Pi \otimes {}^\sigma \eta^{-1}) = L^S(s, {}^\sigma \Pi, \mathrm{Sym}^2 \otimes {}^\sigma \eta^{-1}) \cdot L^S(s, {}^\sigma \Pi, \wedge^2 \otimes {}^\sigma \eta^{-1})$$

has a pole at $s = 1$.

To prove the theorem, we need to show that the Sym^2 L -function does not have a pole at $s = 1$. Suppose for the sake of contradiction that $L^S(s, {}^\sigma \Pi, \mathrm{Sym}^2 \otimes {}^\sigma \eta^{-1})$ has a pole at $s = 1$. Then by Asgari–Shahidi and Hundley–Sayag, one knows that ${}^\sigma \Pi$ is a Langlands functorial transfer from a cuspidal representation of $\mathrm{GSpin}_{2n}(\mathbb{A})$ with central character ${}^\sigma \eta$, and this lift is strong at the archimedean places. Since the archimedean components of ${}^\sigma \Pi$ and ${}^\sigma \eta$ are, by definition, permutations of the archimedean components of Π and η , we deduce that for all archimedean places v , the L-parameter ϕ_v of Π_v factors through the dual group $\mathrm{GSO}_{2n}(\mathbb{C})$ of GSpin_{2n} with similitude character η_v .

As a result, for each archimedean place v , the L-parameter ϕ_v of Π_v preserves both a non-degenerate symmetric bilinear form b_1 and a non-degenerate skew-symmetric bilinear form b_2 on \mathbb{C}^{2n} up to the same similitude character η_v . However, since Π_v is cohomological, for a real place v , it follows from an explicit description of cohomological representations of $\mathrm{GL}_{2n}(\mathbb{R})$, see for example [18, (3.4.1)], that ϕ_v is a direct sum of (2-dimensional) irreducible representations $\phi_{i,v}$ of the Weil group W_{F_v} , and each $\phi_{i,v}$ is not a twist of another $\phi_{j,v}$. This shows that b_1 and b_2 must remain non-degenerate when restricted to each $\phi_{i,v}$. This gives two W_{F_v} -equivariant isomorphisms $\phi_{i,v}^\vee \cong \phi_{i,v} \otimes \eta_v^{-1}$. Since $\phi_{i,v}$ is irreducible, this contradicts Schur’s lemma.

Next, suppose that v is a finite place for which Π_v is supercuspidal. Then essentially the same argument as above goes through. Firstly, by Theorem 4.2, the hypothesis that Π_v has a (η_v, ψ_v) -Shalika period implies that the L-parameter ϕ_v of Π_v factors through $\mathrm{GSp}_{2n}(\mathbb{C})$ with similitude character η_v and is an irreducible $2n$ -dimensional representation of the Weil group W_{F_v} . A result of Henniart [21, 7.4] then says that the L-parameter of ${}^\sigma \Pi_v$ differs from ${}^\sigma \phi_v$ by the (at most quadratic) character $x \mapsto \sigma(|x|^{1/2})/|x|^{1/2}$. But such a twist does not affect irreducibility of the parameter ϕ_v , or the type (symplectic or orthogonal) of an essentially self-dual parameter. Thus the result follows. \square

Remark 4.4. The hypothesis that F is totally real (or that Π has a supercuspidal local component) is rather artificial. One expects the arithmeticity result to hold even without this hypothesis, however, the above proof would not go through. Suppose, for example, F is

an imaginary quadratic extension, then we need to consider cohomological representations of $\mathrm{GL}_{2n}(\mathbb{C})$. For the infinite place v , the parameter of the representation Π_v , which is a $2n$ -dimensional representation of $W_{\mathbb{C}} = \mathbb{C}^{\times}$, is of the form:

$$\phi_v = \bigoplus_{j=1}^{2n} (z \mapsto z^{a_j} \bar{z}^{b_j}).$$

(Here the a_j and b_j are half-integers; see Clozel [6, p.112].) If Π has a Shalika model, then the image of the Langlands parameter ϕ_v is inside a split torus in $\mathrm{Sp}(2n, \mathbb{C})$. But this split torus may also be viewed as sitting inside $\mathrm{SO}(2n, \mathbb{C})$. Hence, from information of $\Pi_{\infty} = \Pi_v$ it is not possible to deduce that the parameter of $\sigma\Pi_v$ is not of orthogonal type.

Remark 4.5. The proof of Theorem 4.3 amounts to showing that

$$L^S(s, \Pi, \wedge^2 \otimes \eta^{-1}) \text{ has a pole at } s = 1 \implies L^S(s, \sigma\Pi, \wedge^2 \otimes \sigma\eta^{-1}) \text{ has a pole at } s = 1$$

under the conditions in theorem. The same proof shows that when F is totally real,

$$L^S(s, \Pi, \mathrm{Sym}^2 \otimes \eta^{-1}) \text{ has a pole at } s = 1 \implies L^S(s, \sigma\Pi, \mathrm{Sym}^2 \otimes \sigma\eta^{-1}) \text{ has a pole at } s = 1$$

when Π is cuspidal cohomological.

5. ARITHMETICITY OF $\mathrm{GL}(n)/F$ -PERIODS FOR REPRESENTATIONS OF $\mathrm{GL}(n)/E$

The argument of the previous section can be applied to prove the arithmeticity of $\mathrm{GL}_n(F)$ -period for representations of $\mathrm{GL}_n(E)$, where E is a quadratic extension of F .

More precisely, let c be the nontrivial element in $\mathrm{Gal}(E/F)$ and let $\omega_{E/F}$ be the quadratic Hecke character associated to E/F by global class field theory. Let χ be a Hecke character of \mathbb{A}_E^{\times} whose restriction to \mathbb{A}_F^{\times} is equal to $\omega_{E/F}$. Then for $\epsilon = \pm$, we set

$$\omega_{E/F}^{\epsilon} = \begin{cases} 1, & \text{if } \epsilon = +; \\ \omega_{E/F} & \text{if } \epsilon = -. \end{cases} \quad \chi^{\epsilon} = \begin{cases} 1 & \text{if } \epsilon = +; \\ \chi & \text{if } \epsilon = -. \end{cases}$$

For a cuspidal representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ and $\epsilon = \pm$, we shall consider the period integral

$$\mathcal{P}^{\epsilon}(\varphi) = \int_{Z_F(\mathbb{A}_F)\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)} \varphi(h) \cdot \omega_{E/F}^{\epsilon}(\det(h)) dh$$

where $\varphi \in \Pi$. For the period integral \mathcal{P}^{ϵ} to have a chance to be nonvanishing, it is necessary that the central character ω_{Π} of Π is equal to $(\omega_{E/F}^{\epsilon})^n$ when restricted to the center $Z_F(\mathbb{A}_F) = \mathbb{A}_F^{\times}$ of $\mathrm{GL}_n(\mathbb{A}_F)$.

Associated to Π is a pair of partial L-functions $L^S(s, \Pi, \mathrm{Asai}^{\pm})$, known as the Asai^{\pm} (or twisted tensor) L-function (see [13, Section 7]). One has

$$L^S(s, \Pi, \mathrm{Asai}^{-}) = L^S(s, \Pi \otimes \chi, \mathrm{Asai}^{+})$$

and

$$L^S(s, \Pi \times \Pi^c) = L^S(s, \Pi, \mathrm{Asai}^{+}) \cdot L^S(s, \Pi, \mathrm{Asai}^{-})$$

where c acts on the representations of $\mathrm{GL}_n(\mathbb{A}_E)$ by

$$\Pi^c(g) = \Pi(c(g)).$$

The following theorem is a consequence of the works of many people (Kim-Krishnamurthy [31, 32], Flicker [10, 11], Ginzburg-Rallis-Soudry [15]).

Theorem 5.1. *For a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$, the following are equivalent:*

- (i) *There is a $\varphi \in \Pi$ such that $\mathcal{P}^c(\varphi) \neq 0$.*
- (ii) *For a sufficiently large finite set S of places of F , the partial Asai $^\epsilon$ L -function $L^S(s, \Pi, \mathrm{Asai}^\epsilon)$ has a pole at $s = 1$.*
- (iii) *$\Pi \otimes \chi^{\epsilon \cdot (-1)^{n-1}}$ is the transfer (standard base change) of a globally generic cuspidal automorphic representation π of the quasi-split $\mathrm{U}_n(\mathbb{A})$.*

One has a local analog of the above global theorem, which is due to the works of many people (A. Kable [28], Anandavardhanan-Kable-Tandon [1], N. Matringe [34, 35]):

Theorem 5.2. *Let v be a non-archimedean place of F which is inert in E and let Π_v be a generic representation of $\mathrm{GL}_n(E_v)$. Then the following are equivalent:*

- (i) *Π_v is $(\mathrm{GL}_n(F_v), \omega_{E_v/F_v}^\epsilon)$ -distinguished.*
- (ii) *The local Asai L -function $L(s, \Pi_v, \mathrm{Asai}^\epsilon)$ has an “exceptional” pole at $s = 0$.*
- (iii) *The L -parameter ϕ_v of Π_v is conjugate-self-dual with sign ϵ (in the sense of [13, Section 3]).*

Observe that this local theorem is more definitive than Theorem 4.2, which is its analog for Shalika periods. In any case, in analogy with Theorem 4.3, one has the following theorem.

Theorem 5.3. *Let Π be a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ which is globally distinguished by $(\mathrm{GL}_n(\mathbb{A}_F), \omega_{E/F}^\epsilon)$. Assume that one of the following conditions hold:*

- (1) *n is odd; or*
- (2) *the number field E is totally real (and hence so is F); or*
- (3) *there is a finite place v of F which is inert in E where Π_v is discrete series.*

Then, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, ${}^\sigma\Pi$ is a cohomological cuspidal representation which is globally distinguished by $(\mathrm{GL}_n(\mathbb{A}_F), \omega_{E/F}^\epsilon)$.

Proof. This is similar to the proof of Theorem 4.3, exploiting Theorems 5.1 and 5.2:

$$\begin{aligned}
& \Pi \text{ is globally distinguished by } (\mathrm{GL}_n(\mathbb{A}_F), \omega_{E/F}^\epsilon) \\
& \implies L^S(s, \Pi, \mathrm{Asai}^\epsilon) \text{ has a pole at } s = 1 \\
& \implies \Pi^c \cong \Pi^\vee \\
& \implies {}^\sigma\Pi^c \cong {}^\sigma\Pi^\vee \\
& \implies L_E^S(s, {}^\sigma\Pi \times {}^\sigma\Pi^c) \text{ has a pole at } s = 1 \\
& \implies L^S(s, {}^\sigma\Pi, \mathrm{Asai}^+) \text{ or } L^S(s, {}^\sigma\Pi, \mathrm{Asai}^-) \text{ has a pole at } s = 1
\end{aligned}$$

Suppose that $L^S(s, \sigma\Pi, \text{Asai}^{-\epsilon})$ has a pole at $s = 1$, rather than $L^S(s, \sigma\Pi, \text{Asai}^{\epsilon})$, so that $\sigma\Pi$ is distinguished by $(\text{GL}_n(\mathbb{A}_F), \omega_{E/F}^{-\epsilon})$. We shall obtain a contradiction under one of the hypotheses (1), (2) or (3).

Under hypothesis (1), so that n is odd, we note that the central character of Π is equal to $\omega_{E/F}^{\epsilon}$ when restricted to the center of $\text{GL}_n(\mathbb{A}_F)$, whereas that of $\sigma\Pi$ is equal to $\omega_{E/F}^{-\epsilon}$. In particular, one restriction is the trivial character of \mathbb{A}_F^{\times} whereas the other is the quadratic character $\omega_{E/F}$. However, at all finite places, it is clear that the central characters of Π_v and $\sigma\Pi_v$ have the same restriction to the center of $\text{GL}_n(F_v)$ since this restriction is at most a quadratic character. This gives the desired contradiction under hypothesis (1).

The argument under hypothesis (2), where E is totally real, is similar as in the proof of Theorem 4.3.

Finally, assume hypothesis (3). For all finite places v of F , $\sigma\Pi_v$ is locally distinguished by $(\text{GL}_n(F_v), \omega_{E_v/F_v}^{-\epsilon})$ and so its L-parameter ϕ'_v is conjugate-self-dual with sign $-\epsilon$. On the other hand, the L-parameter ϕ_v of Π_v is conjugate self-dual of sign ϵ . When Π_v is discrete series, $\phi'_v = \sigma\phi_v$ up to the quadratic character $x \mapsto \sigma(|x|_{E_v}^{1/2})/|x|_{E_v}^{1/2}$ of $W_{E_v}^{ab} \cong E_v^{\times}$. Observe that this character is trivial on F_v^{\times} , so it is conjugate orthogonal in the language of [13]. In particular, ϕ_v and ϕ'_v are conjugate-self-dual of the same sign; this gives the desired contradiction when Π_v is discrete series. □

6. ARITHMETICITY OF GL_{n-1} PERIODS FOR CUSP FORMS ON $\text{GL}_n \times \text{GL}_{n-1}$

We consider a Gross-Prasad like situation, but for groups of type A_n , which was studied by Gan, Gross and Prasad [13]. This context is a very nice generalization of the example in subsection 3.2 where we studied (GL_1, χ) -periods for representations of GL_2 . The non-vanishing of periods is equivalent to a certain central L -value being nonzero. If we further impose the condition that the central value is a critical value then an appropriate algebraicity theorem for this critical value gives arithmeticity. The situation is analogous to Gross's conjecture concerning order of vanishing of critical motivic L -values as discussed in the introduction.

Theorem 6.1. *Let Π be a cohomological cuspidal automorphic representation of $\text{GL}_n(\mathbb{A})$, and say $\Pi \in \text{Coh}(\text{GL}_n, \mu)$. Here \mathbb{A} is the adèle ring of \mathbb{Q} . Similarly, let $\Sigma \in \text{Coh}(\text{GL}_{n-1}, \lambda)$. Suppose that $\Pi \otimes \Sigma$, as a representation of $(\text{GL}_n \times \text{GL}_{n-1})(\mathbb{A})$, has a non-vanishing period with respect to the diagonally embedded subgroup $\text{GL}_{n-1}(\mathbb{A})$. Suppose further that the coefficient systems E_{μ} and E_{λ} satisfy:*

$$\text{Hom}_{\text{GL}_{n-1}}(E_{\mu} \otimes E_{\lambda}, \mathbb{1}) \neq 0.$$

Then for any $\sigma \in \text{Aut}(\mathbb{C})$, the representation $\sigma\Pi \times \sigma\Sigma$ also has a non-vanishing period with respect to $\text{GL}_{n-1}(\mathbb{A})$ under the assumption that [36, Hypothesis 3.10] holds.

Proof. Every step of the proof is a suitable generalization of the proof of arithmeticity of (GL_1, χ) for representations of GL_2 as in subsection 3.2.

To begin, the generalization of Proposition 3.6 goes like this: $\Pi \otimes \Sigma$ as a representation of $(\mathrm{GL}_n \times \mathrm{GL}_{n-1})(\mathbb{A})$ has a non-vanishing $\mathrm{GL}_{n-1}(\mathbb{A})$ period if and only if $L(\frac{1}{2}, \Pi \times \Sigma) \neq 0$. This follows from using the integrals studied by Jacquet, Piatetskii-Shapiro and Shalika [24] as follows. For cusp forms $\phi \in V_\Pi$ and $\phi' \in V_\Sigma$, define

$$\ell(s, \phi, \phi') = \int_{\mathrm{GL}_{n-1}(\mathbb{Q})Z_{n-1}(\mathbb{A}) \backslash \mathrm{GL}_{n-1}(\mathbb{A})} \phi \left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right) \phi'(g) |\det(g)|^{s-\frac{1}{2}} dg,$$

where Z_{n-1} is the center of GL_{n-1} . Our assumption on $\Pi \otimes \Sigma$ is that $\ell(\frac{1}{2}, \phi, \phi') \neq 0$ for some ϕ and ϕ' . Let W_ϕ and $W_{\phi'}$ be the corresponding Whittaker vectors; we may and will take ϕ and ϕ' so that W_ϕ and $W_{\phi'}$ are pure-tensors: $W_\phi = \otimes W_p$ and $W_{\phi'} = \otimes W'_p$. The unfolding argument gives $\ell(s, \phi, \phi') = Z(s, W_\phi, W_{\phi'})$ for $\Re(s) \gg 0$, where

$$Z(s, W_\phi, W_{\phi'}) = \int_{N_{n-1}(\mathbb{A}) \backslash \mathrm{GL}_{n-1}(\mathbb{A})} W_\phi \left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right) W_{\phi'}(g) |\det(g)|^{s-\frac{1}{2}} dg;$$

here N_{n-1} is the subgroup of all upper triangular unipotent elements in GL_{n-1} . The analogue of (3.5) takes the form:

$$(6.2) \quad \ell(s, \phi, \phi') = \left(\prod_{p \in S} \frac{Z(s, W_p, W'_p)}{L_p(s, \Pi_p \otimes \Sigma_p)} \right) \cdot L(s, \Pi \otimes \Sigma).$$

Using [24, Theorem 2.7] we get that both sides and especially both the factors on the right hand side are entire functions. Evaluating at $s = 1/2$ gives

$$(6.3) \quad \Pi \otimes \Sigma \text{ has a non-vanishing } \mathrm{GL}_{n-1}(\mathbb{A}) \text{ period} \iff L(\frac{1}{2}, \Pi \times \Sigma) \neq 0.$$

Next, the hypothesis that $\Pi \in \mathrm{Coh}(\mathrm{GL}_n, \mu)$ and $\Sigma \in \mathrm{Coh}(\mathrm{GL}_{n-1}, \lambda)$ puts us in an arithmetic context, however, this doesn't guarantee that $s = \frac{1}{2}$ is critical. With the foresight of wanting to appeal to algebraicity results, we impose the condition:

$$\mathrm{Hom}_{\mathrm{GL}_{n-1}}(E_\mu \otimes E_\lambda, \mathbb{1}) \neq 0.$$

It was observed by Kasten and Schmidt [29, Theorem 2.3] that this condition implies $s = 1/2$ is critical for the Rankin-Selberg L -function $L(s, \Pi \times \Sigma)$. The same condition is also needed for an algebraicity result for critical values due the second author; see [36]. We get the analogue of (3.8) which looks like

$$(6.4) \quad L(\frac{1}{2}, \Pi \otimes \Sigma) \neq 0 \iff L_f(\frac{1}{2}, \Pi \otimes \Sigma) \neq 0.$$

Under an additional nonvanishing hypothesis involving only representations at infinity as in [36, Hypothesis 3.10], the main result of that paper, [36, Theorem 1.1], says that

$$\sigma \left(\frac{L_f(\frac{1}{2}, \Pi \times \Sigma)}{p^\epsilon(\Pi) p^\eta(\Sigma) \mathcal{G}(\omega_{\Sigma_f}) p_\infty(\mu, \lambda)} \right) = \frac{L_f(\frac{1}{2}, \Pi^\sigma \times \Sigma^\sigma)}{p^\epsilon(\Pi^\sigma) p^\eta(\Sigma^\sigma) \mathcal{G}(\omega_{\Sigma_f^\sigma}) p_\infty(\mu, \lambda)},$$

where $p^\epsilon(\Pi)$ and $p^\eta(\Sigma)$ are nonzero complex numbers, $\mathcal{G}(\omega_{\Sigma_f})$ is the Gauss sum of the central character of Σ , and $p_\infty(\mu, \lambda)$ is a nonzero complex number determined by μ and λ . The analogue of (3.10) follows easily:

$$(6.5) \quad L_f(\frac{1}{2}, \Pi \otimes \Sigma) \neq 0 \iff L_f(\frac{1}{2}, {}^\sigma \Pi \otimes {}^\sigma \Sigma) \neq 0.$$

Arithmeticity follows from first applying (6.3), (6.4) and (6.5) to $\Pi \otimes \Sigma$ and then applying (6.4) and (6.3) to ${}^\sigma \Pi \otimes {}^\sigma \Sigma$ \square

Remark 6.6. The hypothesis on the coefficient systems as in Theorem 6.1, which is itself a nonvanishing period like condition, is crucial for the methods of [36] to apply. Let us note that it is possible to have a pair of cohomological representations Π and Σ for which $s = 1/2$ is critical but for which that condition on the coefficients is not satisfied. For example, take $F = \mathbb{Q}$, $n = 3$, $\mu = (0, 0, 0)$ and $\lambda = (1, -1)$; then E_μ is the trivial representation of GL_3 . Take $\Pi \in \mathrm{Coh}(\mathrm{GL}_3, \mu)$ and $\Sigma \in \mathrm{Coh}(\mathrm{GL}_2, \lambda)$. Then, we leave it to the reader to check that $s = 1/2$ is critical for $L(s, \Pi \times \Sigma)$, but $\mathrm{Hom}_{\mathrm{GL}_2}(E_\mu \otimes E_\lambda, \mathbb{1}) = 0$. Now in such a situation, suppose the representation $\Pi \times \Sigma$ of $\mathrm{GL}_3 \times \mathrm{GL}_2$ has a nonvanishing GL_2 period, then the above proof is not applicable; however, we still believe that one should have arithmeticity.

Remark 6.7. In a certain work in progress [37], the second author is studying algebraicity theorems for critical values of L -functions for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ over any number field. This would then generalize Theorem 6.1 from \mathbb{Q} to any number field.

Remark 6.8. The assumption [36, Hypothesis 3.10] is a certain limitation of the technique used in that paper. We note that this hypothesis is of a purely local nature and depends only the representations Π_∞ and Σ_∞ at infinity. For $n = 2$ the validity of this hypothesis follows from an explicit calculation; see [38]; it is this calculation that gives the term $(2\pi i)^{d_\infty}$ in Proposition 3.9. For $n = 3$ the validity of the hypothesis has been proved by Kasten and Schmidt [29].

7. ARITHMETICITY OF $\mathrm{GL}_n \times \mathrm{GL}_n$ PERIODS FOR CUSP FORMS ON GL_{2n}

In this section, we discuss yet another generalization of the example in subsection 3.2 where we studied (GL_1, χ) -periods for representations π of GL_2 . Indeed, in that example, we could have carried through the entire discussion by replacing π by $\pi \otimes \chi$ and taking the trivial character of $H = \mathrm{GL}_1 \times \mathrm{GL}_1$ sitting as the diagonal torus in GL_2 . (This imposes the condition that the central character of $\pi \otimes \chi$ is trivial.)

Now we take $G = \mathrm{GL}_{2n}$ over a totally real number field F . Take $H = \mathrm{GL}_n \times \mathrm{GL}_n$ sitting as block diagonal matrices in G . Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$ which admits a Shalika model (the analogue of the triviality of the central character mentioned above). We would like to analyze arithmeticity for the periods:

$$\ell(\phi) := \int_{[H]} \phi \left(\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \right) dg_1 dg_2, \quad \phi \in V_\pi$$

where $[H] = Z_G(\mathbb{A}_F)H(F) \backslash H(\mathbb{A}_F)$. Arithmeticity in this context is follows from certain zeta integrals studied by Jacquet-Shalika [25] and Friedberg-Jacquet [12], and an algebraicity result due to Grobner and the second author [18].

Theorem 7.1. *Let π be a cohomological cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ where F is a totally real number field. Suppose that π has a nonvanishing H -period. Further, assume that*

- (1) π admits a Shalika model, and that
- (2) the point $s = \frac{1}{2}$ is critical for $L(s, \pi)$.

Then for any $\sigma \in \text{Aut}(\mathbb{C})$, the representation ${}^\sigma\pi$ also has a non-vanishing H -period.

Proof. Again, we follow the proof of arithmeticity of (GL_1, χ) for representations of GL_2 as in subsection 3.2.

To begin, for a cusp form $\phi \in V_\pi$ define

$$\ell(s, \phi) = \int_{H(F)Z_{2n}(\mathbb{A}_F) \backslash H(\mathbb{A}_F)} \phi \left(\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \right) \left| \frac{\det(g_1)}{\det(g_2)} \right|^{s-\frac{1}{2}} dg_1 dg_2,$$

where Z_{2n} is the center of GL_{2n} . Our assumption on π is that $\ell(\frac{1}{2}, \phi) \neq 0$ for some ϕ . Let \mathcal{S}_ϕ be the corresponding vector in the Shalika model of π ; as before, we may and will take ϕ so that \mathcal{S}_ϕ is a pure-tensor: $\mathcal{S}_\phi = \otimes \mathcal{S}_p$. (For details concerning Shalika models and related matters we refer the reader to [18], and recommend that any serious reader of this section should have that paper by one's side.)

An unfolding argument ([18, Proposition 3.1.5]) gives $\ell(s, \phi) = Z(s, \mathcal{S}_\phi)$ where

$$Z(s, \mathcal{S}_\phi) = \int_{\text{GL}_n(F) \backslash \text{GL}_n(\mathbb{A}_F)} \mathcal{S}_\phi \left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right) |\det(g)|^{s-\frac{1}{2}} dg.$$

The analogue of (6.2) takes the form:

$$(7.2) \quad \ell(s, \phi) = \left(\prod_{p \in S} \frac{Z(s, \mathcal{S}_p)}{L_p(s, \pi_p)} \right) \cdot L(s, \pi).$$

Using [18, Proposition 3.3.1] we get that both sides and both factors on the right hand side are entire functions. Evaluating at $s = 1/2$ gives

$$(7.3) \quad \pi \text{ has a non-vanishing } \text{GL}_n(\mathbb{A}_F) \text{ period} \iff L(\tfrac{1}{2}, \pi) \neq 0.$$

Next, the hypothesis that $\pi \in \text{Coh}(\text{GL}_n, \mu)$ puts us in an arithmetic context, however, as before, this doesn't guarantee that $s = \frac{1}{2}$ is critical. So, we now need the assumption that $s = \frac{1}{2}$ is critical for $L(s, \Pi)$. (In the $\text{GL}_n \times \text{GL}_{n-1}$ case, we needed a stronger condition on the coefficient system, but in the current context [18, Proposition 6.3.1] guarantees that.) The analogue of (6.4) looks like

$$(7.4) \quad L(\tfrac{1}{2}, \pi) \neq 0 \iff L_f(\tfrac{1}{2}, \pi) \neq 0.$$

Under the assumption that $\pi \in \text{Coh}(\text{GL}_{2n}, \mu)$ has a Shalika model, the algebraicity result in [18, Theorem 7.1.2] says

$$\sigma \left(\frac{L_f(\tfrac{1}{2}, \pi \otimes \chi)}{\omega^{\epsilon_\chi}(\pi) \mathcal{G}(\chi) \omega_\infty(\mu)} \right) = \frac{L_f(\tfrac{1}{2}, {}^\sigma\pi \otimes {}^\sigma\chi)}{\omega^{\epsilon_{{}^\sigma\pi}}({}^\sigma\pi) \mathcal{G}({}^\sigma\chi) \omega_\infty(\mu)},$$

where $\omega^{\epsilon_\chi}(\pi)$ is a nonzero complex number, $\mathcal{G}(\chi)$ is the Gauss sum of an algebraic Hecke character χ the parity of which determines ϵ_χ , and $\omega_\infty(\mu)$ is a nonzero complex number determined by μ . The analogue of (6.5) follows easily:

$$(7.5) \quad L_f(\tfrac{1}{2}, \pi) \neq 0 \iff L_f(\tfrac{1}{2}, {}^\sigma\pi) \neq 0.$$

Arithmeticity follows from first applying (7.3), (7.4) and (7.5) to π and then applying (7.4) and (7.3) to ${}^\sigma\pi$. \square

8. ARITHMETICITY FOR CLASSICAL GROUPS

In this section, we consider the possibility of extending some of the results discussed above for $\mathrm{GL}(n)$ to the case of classical groups. By the recent work [2] of Arthur and others, one now has a classification of square-integrable automorphic representations for quasi-split classical groups, in terms of automorphic representations of $\mathrm{GL}(n)$. In view of this, it is natural to ask if arithmeticity results for $\mathrm{GL}(n)$ can be transferred to these classical groups.

More precisely, let G be a symplectic, special orthogonal or unitary group over the number field F and let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$. By Arthur [2], one can attach a discrete A -parameter to π and this is a multiplicity-free formal sum

$$\Psi = \Pi_1 \boxtimes S_{r_1} \oplus \cdots \oplus \Pi_k \boxtimes S_{r_k},$$

where Π_i is a cuspidal automorphic representation of $\mathrm{GL}(n_i)$ (over F or a quadratic extension E) satisfying some symmetry conditions and S_{r_i} is the r_i -dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. Moreover, the set of all π 's with a given discrete A -parameter Ψ is a full near equivalence class in the automorphic discrete spectrum of G . If $r_i = 1$ for all i , Ψ is called a tempered A -parameter.

Now suppose further that π is cohomological. Recall from Proposition 2.4 that for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, there is a square-integrable automorphic representation τ_σ of $G(\mathbb{A}_F)$ such that

$$\tau_{\sigma,f} \cong {}^\sigma \pi_f.$$

Note that, for fixed $\sigma \in \mathrm{Aut}(\mathbb{C})$, τ_σ may not be uniquely determined, but any two such candidates are nearly equivalent to each other and thus have the same A -parameter.

Now one may ask the following questions:

- (1) Does one have a generalization of Theorem 2.5 to the classical group G ? In other words, can one take τ_σ to be ${}^\sigma \pi$? If so, must ${}^\sigma \pi$ be cuspidal?
- (2) Suppose that the A -parameter of π is

$$\Psi = \Pi_1 \boxtimes S_{r_1} \oplus \cdots \oplus \Pi_k \boxtimes S_{r_k},$$

are the Π_i 's necessarily cohomological?

- (3) If the answer to (2) is yes, is the A -parameter of τ_σ (or ${}^\sigma \pi$ if (1) has a positive answer) equal to

$${}^\sigma \Psi = {}^\sigma \Pi_1 \boxtimes S_{r_1} \oplus \cdots \oplus {}^\sigma \Pi_k \boxtimes S_{r_k}?$$

In the remainder of this section, we shall consider these questions for the split group $\mathrm{SO}(2n+1)$.

Theorem 8.1. *Let F be totally real. Let π be a cohomological cuspidal automorphic representation of the split $\mathrm{SO}(2n+1)$ with a tempered A -parameter $\Psi = \Pi_1 \oplus \cdots \oplus \Pi_k$, so that each Π_j is a cuspidal representation of $\mathrm{GL}(2n_j)$ such that $L^S(s, \Pi_j, \wedge^2)$ has a pole at $s = 1$. Then*

- (i) *Each Π_j is a cohomological cuspidal representation of $\mathrm{GL}(2n_j)$.*

(ii) For $\sigma \in \text{Aut}(\mathbb{C})$, let τ_σ be a square-integrable automorphic representation such that $\tau_{\sigma,f} \cong \sigma\pi_f$. Then the A -parameter of τ_σ is ${}^\sigma\Psi = {}^\sigma\Pi_1 \oplus \cdots \oplus {}^\sigma\Pi_k$. In particular, τ_σ is cuspidal and $\tau_{f,\infty}$ belongs to the same L -packet as $\sigma\pi_\infty$.

Proof. (i) For π as in the theorem, the near equivalence class of π (i.e., the global A -packet) contains a globally generic cuspidal representation π_0 (by Arthur [2] and Ginzburg-Rallis-Soudry [15]). In particular, the infinitesimal characters of π_∞ and $\pi_{0,\infty}$ are equal and are integral and “strongly regular”. Thus by Salamanca-Riba [39], we deduce that π_0 is also cohomological. By Gotsbacher–Grobner [16, Proposition 18] (this is where we use the hypothesis that F is totally real), $\pi_{0,\infty}$ is a discrete series representation and has L -parameter of the form

$$\bigoplus_{i=1}^n V(a_i),$$

with $a_1 > a_2 > \cdots > a_n > 0$ and each $a_i \in \frac{1}{2} + \mathbb{Z}$. Here $V(a) = \text{Ind}_{\mathbb{C}^\times}^{W_{\mathbb{R}}}(\chi_a)$ with $\chi_a(z) = (z/\bar{z})^a = z^{2a}/|z\bar{z}|^a$. (See, for example, Gross–Reeder [19, Section 7].) This is also the L -parameter of π_∞ since π_∞ and $\pi_{0,\infty}$ belong to the same L -packet.

With $\Psi = \Pi_1 \oplus \cdots \oplus \Pi_k$, the representations Π_j are regular algebraic; this follows from the description of discrete series parameter as above. It follows from Clozel (see the proof of [6, Théorème 3.13]) that each Π_j is a cohomological cuspidal representation of GL_{2n_j} . This proves (i).

(ii) After (i), ${}^\sigma\Pi_j$ makes sense; see Theorem 2.5. Hence we may consider the A -parameter

$${}^\sigma\Psi = {}^\sigma\Pi_1 \boxplus {}^\sigma\Pi_2 \boxplus \cdots \boxplus {}^\sigma\Pi_r$$

of $\text{GL}(2n)$. Note that, for isobaric sums to have nice rationality properties, we need suitable Tate-twists as in [6, Definition 1.10]. However, the parities of $2n$ and $2n_j$ being all even renders the Tate-twists irrelevant. We observe further that since Π_j has the property that the exterior square L -function has a pole, so does ${}^\sigma\Pi_j$, as we observed in Remark 4.5. Hence, ${}^\sigma\Psi$ is an A -parameter for $\text{SO}(2n+1)$ and gives rise to an associated near equivalence class of square-integrable automorphic representations. We would like to show that the representation τ_σ is contained in this near equivalence class.

To see this, it suffices to consider almost all places v of F where π_v is unramified. At such a place, it is easy to check that unramified local transfer from $\text{SO}(2n+1)$ to $\text{GL}(2n)$ is $\text{Aut}(\mathbb{C})$ -equivariant. Thus τ_σ is nearly equivalent to the representations in the A -packet associated to ${}^\sigma\Psi$, as desired. It follows that $\tau_{\sigma,\infty}$ and $\sigma\pi_\infty$ both belong to the L -packet with L -parameter ${}^\sigma\Pi_{1,\infty} \oplus \cdots \oplus {}^\sigma\Pi_{k,\infty}$. Since $\sigma\pi_\infty$ is discrete series, so is $\tau_{\sigma,\infty}$ and hence τ_σ is necessarily cuspidal. \square

Corollary 8.2. *In the context of the theorem, suppose that $\Psi = \Pi$ is a cuspidal representation of $\text{GL}(2n)$. Then $\sigma\pi$ is a cohomological cuspidal automorphic representation of $\text{SO}(2n+1)$.*

Proof. Under the hypothesis of the corollary, the A -packet associated to Ψ is stable, in the sense that every member of the abstract global A -packet is automorphic. In this case, one may replace $\tau_{\sigma,\infty}$ by $\sigma\pi_\infty$ and thus take τ to be $\sigma\pi$. \square

Remark 8.3. The above proof of the theorem should work for a general classical group G over any number field F , as long as one has the analogue of [16, Proposition 18] for $G(\mathbb{K})$ where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Indeed, assuming (i) of the theorem, the proof of (ii) works in general, using Remark 4.5 for special orthogonal groups and the proof of Theorem 5.3 for unitary groups.

9. ARITHMETICITY OF PERIODS FOR CLASSICAL GROUPS

After Theorem 8.1, it makes sense to consider the question of arithmeticity of periods for classical groups. We consider two examples here.

9.1. Whittaker periods. One may consider the Whittaker period for classical groups. For simplicity, we shall again restrict to the case of $\mathrm{SO}(2n+1)$.

Theorem 9.1. *Let F be a totally real number field, and let $G = \mathrm{SO}(2n+1)$ be the split orthogonal group over F . Let π be a cohomological cuspidal automorphic representation of $G(\mathbb{A}_F)$ which is globally generic. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, the conjugated representation ${}^\sigma\pi$ is also a cohomological cuspidal automorphic representation of $G(\mathbb{A}_F)$ which is globally generic.*

Proof. Since π is globally generic, it follows by [2] and [8] that the A -parameter Ψ of π is tempered. By Theorem 8.1, we know that ${}^\sigma\pi$ belongs to the global A -packet associated to the tempered parameter ${}^\sigma\Psi$. Moreover, ${}^\sigma\pi_v$ is locally generic for all places v . Now in a local L -packet of $\mathrm{SO}(2n+1)$, there can be at most one generic representation by Jiang-Soudry [27]. Thus, ${}^\sigma\pi$ is the only member of its A -packet which could be globally generic. However, the theory of backward lifting [15] says that a tempered A -packet contains a globally generic cuspidal automorphic representation. Thus we conclude that ${}^\sigma\pi$ is a globally generic cohomological cuspidal representation \square

Remark 9.2. We expect the above theorem to hold, with essentially the same proof, for all classical groups G , once Theorem 8.1 (or its appropriate analog) is proved for general G .

9.2. Gross-Prasad period. In this speculative final subsection, we consider the Gross-Prasad periods for the classical groups. To be concrete, let us consider the Gross-Prasad period for unitary groups. Thus, let us suppose the analog of Theorem 8.1 holds for (not necessarily quasi-split) unitary groups over the quadratic extension E/F . Let $\pi = \pi_1 \boxtimes \pi_2$ be a tempered cuspidal representation of $G = \mathrm{U}(n) \times \mathrm{U}(n-1)$. Then a recent preprint of Wei Zhang [47] establishes the global Gross-Prasad conjecture under some local hypotheses. In particular, he shows that the period of π over the diagonally embedded $\mathrm{U}(n-1)$ is nonzero if and only if

$$L_E(\tfrac{1}{2}, BC(\pi_1) \times BC(\pi_2)) \neq 0.$$

Here, $BC(\pi_i)$ denotes the base change of π_i to $\mathrm{GL}(n)$ or $\mathrm{GL}(n-1)$ over E ,

Assume now that π is cohomological, say $\pi \in \mathrm{Coh}(G, \mu)$. Suppose that π has a nonvanishing period over the diagonally embedded $\mathrm{U}(n-1)$, and that $\mathrm{Hom}_{\mathrm{U}(n-1)}(\mu, \mathbb{C}) \neq 0$. Then by the analog of Theorem 8.1 and its corollary, one knows that ${}^\sigma\pi$ is also a cohomological cuspidal automorphic representation of $\mathrm{U}(n) \times \mathrm{U}(n-1)$. Now one may apply the same

argument as in the proof of Theorem 6.1 (with the hypotheses stated there) to deduce that ${}^\sigma\pi$ also has nonvanishing period over the diagonally embedded $U(n-1)$. We will perhaps leave the detailed treatment of this to a future occasion.

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WEE TECK GAN: DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 10 LOWER KENT RIDGE ROAD SINGAPORE 119076

E-mail address: matgwt@nus.edu.sg

A. RAGHURAM: INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH (IISER), FIRST FLOOR,
CENTRAL TOWER, SAI TRINITY BUILDING GARWARE CIRCLE, SUTARWADI, PASHAN PUNE, MAHARASHTRA
411021, INDIA.

E-mail address: `raghuram@iiser.pune.ac.in`